Final exam — Ordinary Differential Equations (WIGDV-07)

Thursday 29 January 2015, 9.00h–12.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Question 1 (10 points)

Solve the following initial value problem:

$$\frac{dy}{dx} = \frac{y}{x} + \tan\left(\frac{y}{x}\right), \qquad y(1) = \frac{\pi}{4}.$$

What is the largest interval on which the solution exists?

Question 2 (10 points)

Solve the following Bernoulli equation:

$$\frac{dy}{dx} = -\frac{1}{x}y + \sqrt{y}, \qquad x > 0.$$

Question 3 (10 points)

Use an integrating factor of the form $M(x, y) = x^\alpha y^\beta$ to solve the following equation:

$$(2y^2 + 5x^3y)\,dx + (4xy + 3x^4)\,dy = 0.$$

Question 4 (3 + 12 points)

(a) Give the definition of "a **fundamental matrix** for a homogeneous $n \times n$ linear system of differential equations."

(b) Compute a fundamental matrix for the following system:

$$\frac{d\mathbf{y}}{dt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \mathbf{y}.$$

Question 5 (3 + 12 + 5 points)

(a) Formulate Banach's fixed point theorem.

(b) Let $C([0,1])$ be the space of continuous real-valued functions on the interval $[0,1]$ which is equipped with the norm

$$\|y\| = \sup_{x\in[0,1]} |y(x)|.$$

Consider the integral operator

$$T : C([0,1]) \to C([0,1]), \quad (Ty)(x) = \eta + \int_0^x t \arctan(y(t))\, dt.$$

Prove that for all $y, z \in C([0,1])$ we have

$$\|Ty - Tz\| \leq \tfrac{1}{2}\|y - z\|.$$

(c) Prove that the initial value problem

$$\frac{dy}{dx} = x \arctan(y), \quad y(0) = \eta$$

has a unique solution in the space $C([0,1])$. You may use without proof that $C([0,1])$ is a Banach space.

Question 6 (10 points)

Compute the general solution of the following 3rd order equation:

$$u''' - 5u'' + 9u' - 5u = -5x^2 + 8x - 7.$$

Question 7 (10 + 5 points)

Consider the semi-homogeneous boundary value problem

$$x^2u'' + 2xu' = f(x), \qquad u(1) = 0, \qquad u(2) = 0,$$

where $f(x)$ is a continuous function.

(a) Compute Green's function.
Hint: the homogeneous differential equation has solutions of the form $u = x^\lambda$.

(b) Use Green's function to solve the boundary value problem for $f(x) = 2x$.

End of test (90 points)

Solution question 1 (10 points)

- The variable $u = y/x$ satisfies a differential equation with separated variables:

$$\frac{du}{dx} = \frac{\tan u}{x} \quad \Rightarrow \quad \int \frac{1}{\tan u}\, du = \int \frac{1}{x}\, dx \quad \Rightarrow \quad \int \frac{\cos u}{\sin u}\, du = \int \frac{1}{x}\, dx.$$

  (2 points)

- Working out the integrals gives

$$\log|\sin u| = \log|x| + C \quad \Rightarrow \quad \sin u = Kx \quad \Rightarrow \quad u = \arcsin(Kx),$$

  where $K = \pm e^C$. Hence, the general solution is given by

$$y = x\arcsin(Kx).$$

  (4 points)

- The initial condition $y(1) = \pi/4$ implies that $K = 1/\sqrt{2}$.
  (2 points)

- The function $\arcsin(x)$ is defined on the closed interval $[-1, 1]$. Therefore, the solution of the initial value problem is defined on the closed interval $[-1/K, 1/K] = [-\sqrt{2}, \sqrt{2}]$.
  (2 points)

Solution question 2 (10 points)

- Since the exponent of the nonlinear term is $\alpha = \frac{1}{2}$ we define the new variable $z = y^{1-\alpha} = \sqrt{y}$ which satisfies a linear differential equation:

$$z' + \frac{1}{2x}z = \frac{1}{2}.$$

  (3 points)

- Multiplying the equation with the integrating factor $\phi(x) = \sqrt{x}$ gives

$$\sqrt{x}z' + \frac{1}{2\sqrt{x}}z = \frac{1}{2}\sqrt{x} \quad \Leftrightarrow \quad \frac{d}{dx}\left[\sqrt{x}z\right] = \frac{1}{2}\sqrt{x} \quad \Leftrightarrow \quad z = \frac{x}{3} + \frac{C}{\sqrt{x}}.$$

  (5 points)

- Hence, the solution of Bernoulli's equation is given by

$$y = z^2 = \left(\frac{x}{3} + \frac{C}{\sqrt{x}}\right)^2.$$

  (2 points)

Remark. The linear differential equation for z can also be solved by first solving the homogeneous equation and then applying variation of constants to find a particular solution.

Solution question 3 (10 points)

- After multiplying with $M(x,y) = x^\alpha y^\beta$ the equation reads as

$$\underbrace{\left(2x^\alpha y^{\beta+2} + 5x^{\alpha+3}y^{\beta+1}\right)}_{g}\,dx + \underbrace{\left(4x^{\alpha+1}y^{\beta+1} + 3x^{\alpha+4}y^{\beta}\right)}_{h}\,dy = 0.$$

  The equation is exact if and only if

$$\begin{aligned} g_y = h_x \quad &\Leftrightarrow \quad 2(\beta+2)x^\alpha y^{\beta+1} + 5(\beta+1)x^{\alpha+3}y^\beta = 4(\alpha+1)x^\alpha y^{\beta+1} + 3(\alpha+4)x^{\alpha+3}y^\beta \\ &\Leftrightarrow \quad 2(\beta+2) = 4(\alpha+1) \quad \text{and} \quad 5(\beta+1) = 3(\alpha+4) \\ &\Leftrightarrow \quad \alpha = 1 \quad \text{and} \quad \beta = 2. \end{aligned}$$

  Therefore, the integrating factor is given by $M(x,y) = xy^2$.
  (4 points)

- Next we want to find a potential function. Define

$$F(x,y) = \int g(x,y)\,dx + \phi(y) = \int 2xy^4 + 5x^4y^3\,dx + \phi(y) = x^2y^4 + x^5y^3 + \phi(y).$$

  By construction we satisfy $F_x = g$. The equation $F_y = h$ is satisfied if and only if $\phi'(y) = 0$. For example, we can just take $\phi(y) = 0$.
  (4 points)

- The solution of the differential equation is given by the implicit equation

$$F(x,y) = C \quad \Leftrightarrow \quad x^2y^4 + x^5y^3 = C.$$

  where C is an arbitrary constant.
  (2 points)

Solution question 4 (3 + 12 points)

(a) An $n \times n$ matrix $Y(t)$ is a fundamental matrix for an $n \times n$ linear system $\mathbf{y}' = A(t)\mathbf{y}$ if it has the following properties:

  (i) The columns of $Y(t)$ are solutions of the differential equation. (Equivalent statement: $Y'(t) = A(t)Y(t)$.)

  (ii) The columns of $Y(t)$ are linearly independent. (Equivalent statement: $Y(t)$ is invertible.)

  (3 points)

(b)
- The coefficient matrix and its characteristic polynomial are given by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \quad \Rightarrow \quad \det(A - \lambda I) = (1-\lambda)(\lambda^2 - 2\lambda + 1) = (1-\lambda)^3.$$

  Hence, $\lambda = 1$ is the only eigenvalue of A with multiplicity three.
  (2 points)

- Straightforward calculations show that

$$A - I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (A-I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the first two (generalized) eigenspaces of A are given by

$$E^1_\lambda = \mathrm{Nul}(A-I) = \ \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$E^2_\lambda = \mathrm{Nul}(A-I)^2 = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

The dot diagram associated with the eigenvalue $\lambda = 1$ is given by

$$\begin{aligned} r_1 &= \dim E^1_\lambda = 2 \\ r_2 &= \dim E^2_\lambda - \dim E^1_\lambda = 3 - 2 = 1 \end{aligned} \quad \Rightarrow \quad \begin{matrix} \bullet & \bullet \\ \bullet & \end{matrix}$$

This means that we have one cycle of length 2 and one cycle of length 1. **(4 points)**

- The 1-cycle of is just a vector $\mathbf{v} \in E^1_\lambda$. For example, we can choose

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The 2-cycle of length 2 is given by $\{(A-I)\mathbf{w}, \mathbf{w}\}$ where $\mathbf{w} \in E^2_\lambda \setminus E^1_\lambda$. For example, we can choose

$$\mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad (A-I)\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

(2 points)

- If we choose to list the 1-cycle first, then the Jordan canonical form becomes $A = QJQ^{-1}$ with

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(2 points)

- A possible fundamental matrix is given by $Y(t) = e^{At} = Qe^{Jt}Q^{-1}$. Observe that $Z(t) = e^{At}Q = Qe^{Jt}$ is also a fundamental matrix. (Recall that fundamental matrices can always be multiplied with an invertible matrix on the right hand side.) Choosing the latter avoids the computation of Q^{-1} which gives

$$Z(t) = Qe^{Jt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix} = e^t \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 1 & 1+t \end{bmatrix}.$$

(2 points)

Remark. Part (b) can also be solved without the Jordan canonical form. We can write $A = I + N$ where

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}.$$

It is obvious that $IN = NI$ which implies that we can use the rule $e^{At} = e^{It}e^{Nt}$. Moreover, the matrix N is nilpotent because $N^3 = 0$. Therefore, $e^{Nt} = I + Nt + \frac{1}{2}N^2t^2$.

Note, however, that the decomposition $A = D + M$ where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

does not work. The main reason is that $DM \neq MD$. Hence, the rule $e^{At} = e^{Dt}e^{Mt}$ can *not* be applied! Also observe that M is not nilpotent, which makes the computation of e^{Mt} somewhat harder because the infinite series does not reduce to a finite sum.

Solution question 5 (3 + 12 + 5 points)

(a) Let D be a closed nonempty subset in a Banach space B. Let the operator $T : D \to B$ map D into itself, i.e., $T(D) \subset D$, and be a contraction: there exists a number $0 < q < 1$ such that

$$\|Tx - Ty\| \leq q\|x - y\|, \qquad \forall\, x, y \in D,$$

Then the fixed point equation $Tx = x$ has precisely one solution $\bar{x} \in D$. Moreover, iterations of T converge to this fixed point:

$$x_0 \in D, \quad x_{n+1} = Tx_n \quad \Rightarrow \quad \lim_{n\to\infty} x_n = \bar{x}.$$

(3 points)

(b)
- The mean value theorem implies that for all $y, z \in \mathbb{R}$ there exists a number $u \in \mathbb{R}$ between y and z such that

$$\arctan(y) - \arctan(z) = \arctan'(u)(y - z) = \frac{1}{1 + u^2}(y - z).$$

(2 points)

- Hence, for all $y, z \in C([0, 1])$ and $t \in [0, 1]$ we have

$$|\arctan(y(t))) - \arctan(z(t)))| \leq |y(t) - z(t)|.$$

(2 points)

- For all $x \in [0, 1]$ we have

$$\begin{aligned} |(Ty)(x) - (Tz)(x)| &= \left| \int_0^x t\big[\arctan(y(t)) - \arctan(z(t))\big]\, dt \right| \\ &\leq \int_0^x t|\arctan(y(t)) - \arctan(z(t))|\, dt \\ &\leq \int_0^x t|y(t) - z(t)|\, dt. \end{aligned}$$

(4 points)

- Since $|y(t) - z(t)| \leq \|y - z\|$ for all $t \in [0,1]$ we obtain

$$|(Ty)(x) - (Tz)(x)| \leq \int_0^x t\,dt \|y - z\| = \tfrac{1}{2}x^2\|y - z\| \leq \tfrac{1}{2}\|y - z\|.$$

(2 points)

- Since this inequality holds for all $x \in [0,1]$ we can take the supremum on the left hand side to obtain

$$\|Ty - Tz\| \leq \tfrac{1}{2}\|y - z\|.$$

(2 points)

(c)
- Applying Banach's fixed point theorem with

$$B = D = C([0,1]), \qquad T : B \to B, \qquad (Ty)(x) = \eta + \int_0^x t \arctan(y(t))\,dt$$

shows that there exists precisely one function $\bar{y} \in C([0,1])$ such that

$$\bar{y}(x) = \eta + \int_0^x t \arctan(\bar{y}(t))\,dt.$$

(2 points)

- Moreover, satisfying the latter equation is equivalent to satisfying the initial value problem. This proves that the initial value problem has a unique solution in the space $C([0,1])$.
(3 points)

Solution question 6 (10 points)

- The characteristic polynomial associated with the homogeneous differential equation is given by

$$\lambda^3 - 5\lambda^2 + 9\lambda - 5 = (\lambda - 1)(\lambda^2 - 4\lambda + 5) = (\lambda - 1)((\lambda - 2)^2 + 1).$$

The zeros of this polynomial are $\lambda = 1$ and $\lambda = 2 \pm i$.
(3 points)

- The general solution in complex form is given by

$$u_h = c_1 e^x + c_2 e^{(2+i)x} + c_3 e^{(2-i)x}.$$

The general solution in real form is given by

$$u_h = d_1 e^x + d_2 e^{2x}\cos(x) + d_3 e^{2x}\sin(x).$$

Both solutions are accepted.
(1 point)

- For the particular solution we try a quadratic polynomial:

$$u_p = Ax^2 + Bx + C \quad \Rightarrow \quad u_p' = 2Ax + B \quad \Rightarrow \quad u_p'' = 2A.$$

(1 point)

- Substitution in the nonhomogeneous equation gives

$$-5Ax^2 + (18A - 5B)x - 10A + 9B - 5C = -5x^2 + 8x - 7.$$

Equating like powers of x on both sides gives the following system of equations:

$$\begin{bmatrix} -5 & 0 & 0 \\ 18 & -5 & 0 \\ -10 & 9 & -5 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \\ 7 \end{bmatrix},$$

which has the unique solution $A = 1$, $B = 2$, and $C = 3$.
(4 points)

- Hence, the general solution of the differential equation is given by

$$u = u_h + u_p = c_1 e^x + c_2 e^{(2+i)x} + c_3 e^{(2-i)x} + x^2 + 2x + 3,$$

or, equivalently,

$$u = u_h + u_p = d_1 e^x + d_2 e^{2x} \cos(x) + d_3 e^{2x} \sin(x) + x^2 + 2x + 3.$$

(1 point)

Solution question 7 (10 + 5 points)

(a)

- The associated differential operator is given by

$$L = p(x)\frac{d^2}{dx^2} + p'(x)\frac{d}{dx} + q(x),$$

where $p(x) = x^2$ and $q(x) = 0$. Substituting $u = x^\lambda$ in the homogeneous differential equation $Lu = 0$ gives the characteristic equation

$$\lambda(\lambda - 1) + 2\lambda = 0 \quad \Leftrightarrow \quad \lambda^2 + \lambda = 0 \quad \Leftrightarrow \quad \lambda(\lambda + 1) = 0.$$

Hence, the general solution of the homogeneous differential equation is given by $u = a + b/x$.
(3 points)

- Next, we have to choose one function u_1 that satisfies the left boundary condition $u(1) = 0$ and one function u_2 that satisfies the right boundary condition $u(2) = 0$. For example, we can take

$$u_1 = 1 - \frac{1}{x} \qquad \text{and} \qquad u_2 = 1 - \frac{2}{x}.$$

(2 points)

- Their Wronskian is given by

$$W = \det \begin{bmatrix} u_1 & u_2 \\ u_1' & u_2' \end{bmatrix} = u_1 u_2' - u_1' u_2 = \frac{1}{x^2}.$$

(2 points)

- Now we have all the ingredients to compute Green's function:

$$\Gamma(x,\xi) = \frac{1}{W(\xi)p(\xi)} \cdot \begin{cases} u_1(\xi)u_2(x) & \text{if } 1 \leq \xi \leq x \leq 2 \\ u_1(x)u_2(\xi) & \text{if } 1 \leq x \leq \xi \leq 2 \end{cases}$$

$$= \begin{cases} \left(1 - \frac{1}{\xi}\right)\left(1 - \frac{2}{x}\right) & \text{if } 1 \leq \xi \leq x \leq 2 \\ \left(1 - \frac{1}{x}\right)\left(1 - \frac{2}{\xi}\right) & \text{if } 1 \leq x \leq \xi \leq 2 \end{cases}$$

(3 points)

(b)

- With Green's function the boundary value problem can be solved by computing the integral

$$u(x) = \int_1^2 \Gamma(x,\xi)f(\xi)\,d\xi.$$

(1 point)

- Substituting Green's function, $f(\xi) = 2\xi$, and splitting the integrals gives

$$\begin{aligned} u(x) &= \left(1 - \frac{2}{x}\right)\int_1^x 2\xi - 2\,d\xi + \left(1 - \frac{1}{x}\right)\int_x^2 2\xi - 4\,d\xi \\ &= \left(1 - \frac{2}{x}\right)(x^2 - 2x + 1) + \left(1 - \frac{1}{x}\right)(-x^2 + 4x - 4) \\ &= -3 + \frac{2}{x} + x. \end{aligned}$$

(4 points)